Apply Kalman Filter in Financial Time Series

Final Project for EE616 Signal Detection & Estimation

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Introduction

- Financial time series are well-known non-stationary.
- There's no perfect prediction model for such time series.
- A fundamental assumption is that the underlying series are driven by some hidden control or variables.
- A good approximate model should,
  - demonstrates the hidden effects (state-space model)
  - provide a good prediction performance (mean square error)
  - computationally efficient (recursive filtering)
- In this project, I will use dynamic state-space system to model the financial time series, and then use Kalman filter to efficiently make prediction.

Review Kalman Filter

- Under a Gaussian-Markov state model ($\mathbf{u}[n] \sim \mathcal{N}(0, \mathbf{Q})$)

$$\mathbf{s}[n] = \mathbf{A}\mathbf{s}[n-1] + \mathbf{B}\mathbf{u}[n]$$

- and Bayesian linear observation model ($\mathbf{w}[n] \sim \mathcal{N}(0, \mathbf{C}[n])$)

$$\mathbf{x}[n] = \mathbf{H}[n]\mathbf{s}[n] + \mathbf{w}[n]$$

- a Kalman filter is a recursive (prediction & correction only use present input $\mathbf{x}[n]$ and previous output $\hat{\mathbf{s}}[n-1|n-1]$, $\mathbf{K}[n]$ is Kalman Gain [1])

$$\begin{aligned} \hat{\mathbf{s}}[n|n-1] &= \mathbf{A}\hat{\mathbf{s}}[n-1|n-1] \\ \hat{\mathbf{s}}[n|n] &= \hat{\mathbf{s}}[n|n-1] + \mathbf{K}[n](\mathbf{x}[n] - \mathbf{H}[n]\hat{\mathbf{s}}[n|n-1]) \end{aligned}$$

- MMSE estimator (**M** are minimum mean square error matrix).

$$\begin{aligned} \mathbf{M}[n|n-1] &= \mathbf{A}\mathbf{M}[n-1|n-1]\mathbf{A}^T + \mathbf{B}\mathbf{Q}\mathbf{B}^{-1} \\ \mathbf{M}[n|n] &= (\mathbf{I} - \mathbf{K}[n]\mathbf{H}[n])\mathbf{M}[n|n-1] \end{aligned}$$

[1] $\mathbf{K}[n] = \mathbf{M[n|n-1]H^T[n](C[n] + H[n]M[n|n-1]H^T[n])^{-1}}$

Basic Model

In finance, compare to the assets price p, the rate of return r tend to behavior more stationary. We denote r as

$$r[n] = \log(p[n]) - \log(p[n-1])$$

Although the true value of $r[n]$ is unknown, we could always observer it in noise market by,

$$R[n] = r[n] + w[n] \qquad w[n] \sim \mathcal{N}(0, \sigma_w^2)$$

In this project, I will analyzing two models with different assumptions as follows

- r is constant.
- r is mean reverting.

Constant r

We firstly assume the r is constant, then,

$$r[n] = r[n-1] + u$$

We further assume the observation and process noises are WSS ($u \sim \mathcal{N}(0, \sigma_u^2)$, $w \sim \mathcal{N}(0, \sigma_w^2)$) and $\sigma_u^2 \ll \sigma_w^2$.
Recall the Kalman filter discussion, we have

$$\begin{aligned}
\hat{r}[n|n-1] &= \hat{r}[n-1|n-1] \\
M[n|n-1] &= M[n-1|n-1] + \sigma_u^2 \\
K[n] &= \frac{M[n|n-1]}{\sigma_w^2 + M[n|n-1]} \\
\hat{r}[n|n] &= \hat{r}[n|n-1] + K[n](R[n] - \hat{r}[n|n-1]) \\
M[n|n] &= (1 - K[n])M[n|n-1]
\end{aligned}$$

Constant r - Parameter Estimation

- In the above model, we assume σ_u, σ_w and μ are constant parameters.
- Now we estimate them from real data.
- Recall the Gaussian Linear assumption and $\sigma_u^2 \ll \sigma_w^2$,

$$\begin{aligned} \mathbf{R} &\sim \mathcal{N}(r, \sigma_w^2 \mathbf{I}) \\ \mathbf{r} &\sim \mathcal{N}(\mu, \sigma_u^2 \mathbf{I}) \\ \mathbf{R} &\sim \mathcal{N}(\mu, (\sigma_w^2 + \sigma_u^2)\mathbf{I}) \\ \mathbf{R} &\sim \mathcal{N}(\mu, \sigma_w^2 \mathbf{I}) \end{aligned}$$

- The MLE of $\gamma = \begin{bmatrix} \sigma_w & \mu \end{bmatrix}^T$ is given by,

$$\arg\max_{\gamma} L(\gamma | \mathbf{R})$$

Constant r - Parameter Estimation

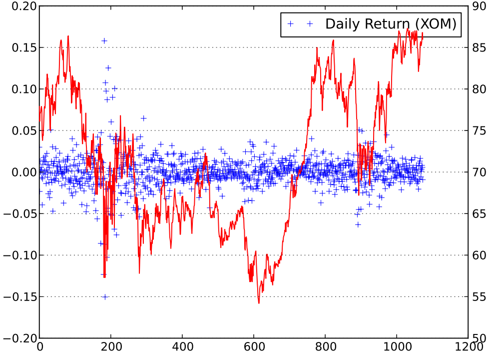
$$
\begin{aligned}
\mathbf{R} &\sim \mathcal{N}(\mu, \sigma_w^2 \mathbf{I}) \\
\log L(\gamma|\mathbf{R}) &= \log f(\mathbf{R}, \gamma) \\
&= \log \frac{\exp(-\frac{\sum_n (R[n]-\mu)^2}{2\sigma_w^2})}{(2\pi\sigma_w)^{N/2}} \\
&= \frac{N}{2}\log(2\pi\sigma_w^2) + \frac{\sum_n (R[n]-\mu)^2}{2\sigma_w^2} \\
\frac{\partial \log L(\gamma|\mathbf{R})}{\partial \mu} &= \frac{\sum_n (R[n]-\mu)}{\sigma_w^2} \qquad \text{(set to 0)} \\
\hat{\mu} &= \frac{1}{N}\sum_n R[n] \\
&= \bar{R}[n] \qquad \text{(MLE of } \mu)
\end{aligned}
$$

Constant r - Parameter Estimation

$$
\begin{aligned}
\frac{\partial \log L(\gamma|\mathbf{R})}{\partial \sigma_w^2} &= \frac{N}{2\sigma_w^2} - \frac{\sum_n (R[n]-\mu)^2}{2\sigma_w^4} && \text{(set to 0)} \\
\hat{\sigma}_w^2 &= \frac{1}{N}\sum_n (R[n]-\mu)^2 \\
&= \frac{1}{N}\sum_n (R[n]-\bar{R}[n])^2 && \text{(MLE of } \sigma_w^2\text{)} \\
\hat{\gamma} &= \begin{bmatrix} \bar{R}[n] \\ \frac{1}{N}\sum_n (R[n]-\bar{R}[n])^2 \end{bmatrix}
\end{aligned}
$$

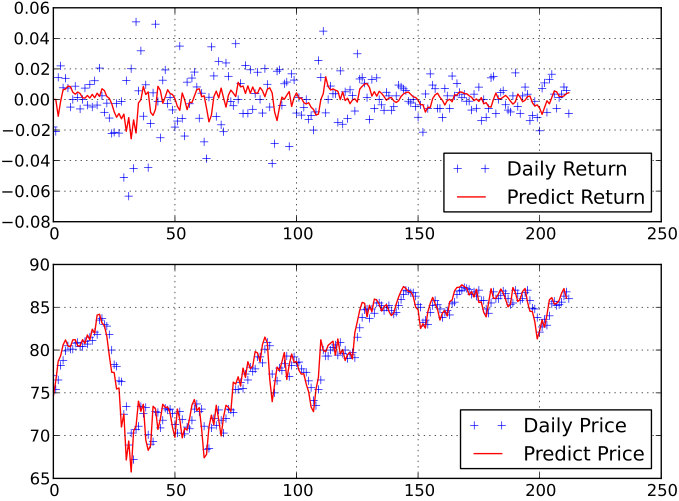
Constant Model - Application

- Exxon Mobil Corporation(NYSE:XOM) historical daily price and return from 2008-01-23 to 2012-04-26.
- Use first 80% data to find the MLE of $\gamma = \begin{bmatrix} 0.003936\% & 0.0435\% \end{bmatrix}^T$.



Constant Model - Application

- Use latest 20% data to recursively evaluate the $\hat{r}[n|n-1]$.



Mean-reverting Model

- Now we relax r's constant assumption.
- Let us assume $\mathrm{E}(r_n) = \mu$, and r is mean-reverting.

$$r_n - r_{n-1} = \alpha(\mu - r_{n-1}) + u$$

- Then the state space model will be given by

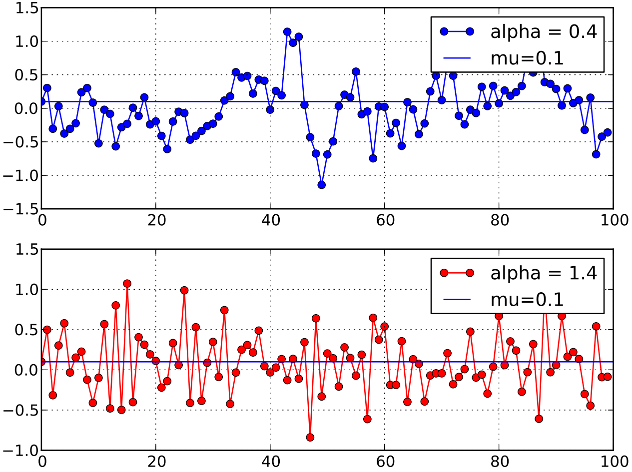
$$\begin{aligned} r_n &= (1-\alpha) r_{n-1} + \alpha\mu + u \\ \mathrm{E}(r) &= \mu \\ \mathrm{var}(r) &= \frac{\sigma_u^2}{2\alpha - \alpha^2} \end{aligned}$$

- The observation model will be given by

$$R_n = r_n + w$$

Mean-reverting Model - 100 sample simulation

$$\mu = 0.1 \quad \sigma_u^2 = 0.1$$



Mean-reverting Model - Parameter Estimation

- In the above model, we assume the α, σ_w, σ_u and μ are unknown constant parameters.
- According to the linear Gaussian assumption,

$$\begin{aligned} R_n &= r_n + w \\ &= (1-\alpha)r_{n-1} + \alpha\mu + u + w \\ &= (1-\alpha)(R_{n-1} - w) + \alpha\mu + u + w \\ &= (1-\alpha)R_{n-1} + \alpha\mu + u + \alpha w \end{aligned}$$

  which shows R_n is an autoregressive process AR(1).
- We would like to obtain the MLE of $\boldsymbol{\gamma} = \begin{bmatrix}\alpha & \sigma_w^2 & \sigma_u^2 & \mu\end{bmatrix}^T$,

$$\arg\max_{\boldsymbol{\gamma}} L(\boldsymbol{\gamma}|\mathbf{R})$$

Mean-reverting Model - Conditional MLE

$$R_n | R_{n-1} \sim \mathcal{N}((1-\alpha)R_{n-1} + \alpha\mu, \sigma_u^2 + \alpha^2\sigma_w^2)$$

$$f(R_n | R_{n-1}, \gamma)$$
$$= \frac{1}{\sqrt{2\pi(\sigma_u^2 + \alpha^2\sigma_w^2)}} \exp(-\frac{(R_n - (1-\alpha)R_{n-1} - \alpha\mu)^2}{2(\sigma_u^2 + \alpha^2\sigma_w^2)})$$

$$\log(f(R_n | R_{n-1}, \gamma))$$
$$= -\frac{\log(2\pi(\sigma_u^2 + \alpha^2\sigma_w^2))}{2} - \frac{(R_n - (1-\alpha)R_{n-1} - \alpha\mu)^2}{2(\sigma_u^2 + \alpha^2\sigma_w^2)}$$

Mean-reverting Model - Marginal MLE

Recall **R** is a stationary AR(1) process, we can assume R_1 as,

$$E[R_1] = \mu \qquad \mathrm{var}[R_1] = \frac{\sigma_u^2 + \alpha^2\sigma_w^2}{2\alpha - \alpha^2}$$

$$R_1 \sim \mathcal{N}\left(\mu, \frac{\sigma_u^2 + \alpha^2\sigma_w^2}{2\alpha - \alpha^2}\right)$$

$$f(R_1, \gamma) = \left(\frac{2\pi(\sigma_u^2 + \alpha^2\sigma_w^2)}{2\alpha - \alpha^2}\right)^{-1/2} \exp\left(-\frac{(R_1 - \mu)^2(2\alpha - \alpha^2)}{2(\sigma_u^2 + \alpha^2\sigma_w^2)}\right)$$

$$\log(f(R_1, \gamma)) = -\frac{1}{2}\log\left(\frac{2\pi(\sigma_u^2 + \alpha^2\sigma_w^2)}{2\alpha - \alpha^2}\right) - \frac{(R_1 - \mu)^2(2\alpha - \alpha^2)}{2(\sigma_u^2 + \alpha^2\sigma_w^2)}$$

Mean-reverting Model - Exact MLE

$$f(R_n, \ldots, R_1|\gamma) = f(R_1, \gamma) \prod_{t=2}^{n} f(R_n|R_{n-1}, \gamma)$$

$$\begin{aligned}
\log L(\gamma|\mathbf{R}) &= \log f(R_1, \gamma) + \sum_{t=2}^{n} \log f(R_t|R_{t-1}, \gamma) \\
&= -\frac{1}{2} \log \left(\frac{2\pi(\sigma_u^2 + \alpha^2 \sigma_w^2)}{2\alpha - \alpha^2} \right) - \frac{(R_1 - \mu)^2 (2\alpha - \alpha^2)}{2(\sigma_u^2 + \alpha^2 \sigma_w^2)} \\
&\quad - \sum_{t=2}^{n} \left(\frac{\log(2\pi(\sigma_u^2 + \alpha^2 \sigma_w^2))}{2} + \frac{(R_t - (1-\alpha)R_{t-1} - \alpha\mu)^2}{2(\sigma_u^2 + \alpha^2 \sigma_w^2)} \right) \\
&= \frac{\log(2\alpha - \alpha^2)}{2} - \frac{n}{2} \log(2\pi(\sigma_u^2 + \alpha^2 \sigma_w^2)) - \frac{(R_1 - \mu)^2 (2\alpha - \alpha^2)}{2(\sigma_u^2 + \alpha^2 \sigma_w^2)} \\
&\quad - \frac{1}{2(\sigma_u^2 + \alpha^2 \sigma_w^2)} \sum_{t=2}^{n} (R_t - (1-\alpha)R_{t-1} - \alpha\mu)^2
\end{aligned}$$

Mean-reverting Model - Kalman filter

Notice that the log-likelihood function $\log L(\gamma|\mathbf{R})$ is a non-linear function, so there's no exact analytical solution for MLE $\hat{\gamma}$. here we use numerical method,

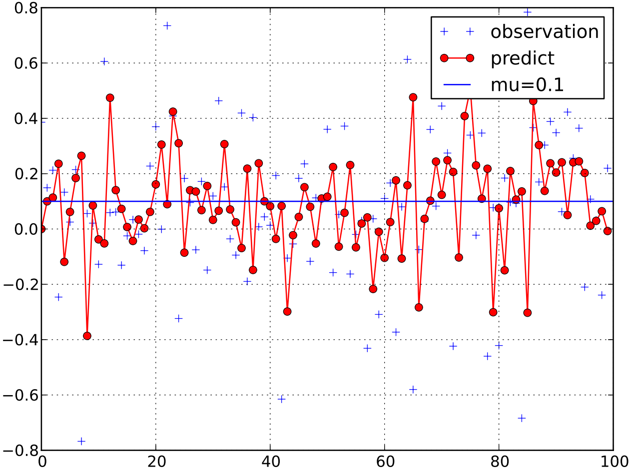
$$\arg\max_{\gamma} \log L(\gamma|\mathbf{R})$$

We then use MLE γ to configure a Kalman filter.

$$\begin{aligned}
\hat{r}[n|n-1] &= (1-\hat{\alpha})\hat{r}[n-1|n-1] + \hat{\alpha}\hat{\mu} \\
M[n|n-1] &= (1-\hat{\alpha})^2 M[n-1|n-1] + \hat{\sigma}_u^2 \\
K[n] &= \frac{M[n|n-1]}{\hat{\sigma}_w^2 + M[n|n-1]} \\
\hat{r}[n|n] &= \hat{r}[n|n-1] + K[n](R[n] - \hat{r}[n|n-1]) \\
M[n|n] &= (1-K[n])M[n|n-1]
\end{aligned}$$

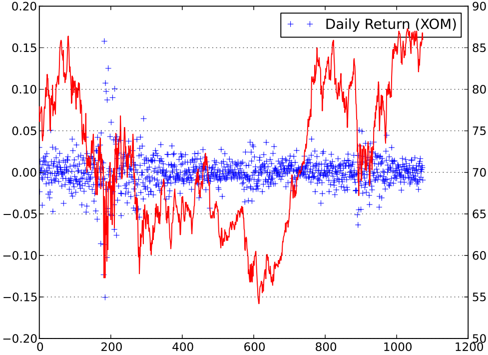
Mean-reverting Model - 100 sample simulation

$$\mu = 0.1 \quad \sigma_u^2 = 0.1 \quad \sigma_w^2 = 0.01 \quad \alpha = 1.4$$



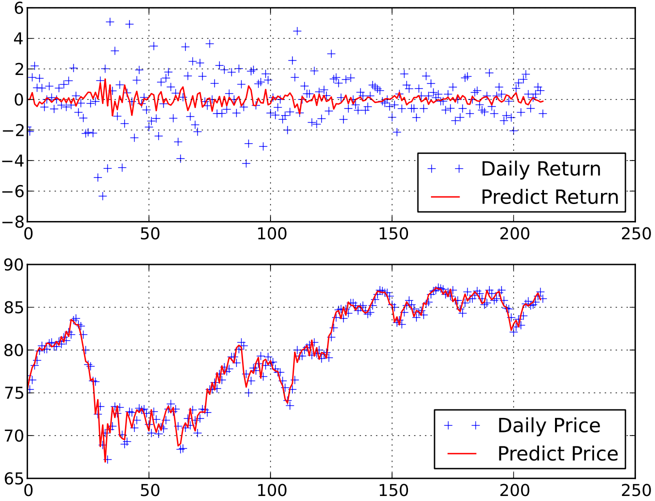
Mean-reverting Model - Application

- Exxon Mobil Corporation(NYSE:XOM) historical daily price and return from 2008-01-23 to 2012-04-26.
- Use first 80% data to find the MLE of $\gamma = \begin{bmatrix} 1.211 & 5.66 \times 10^{-4}\% & 4.15\% & 3.17 \times 10^{-3}\% \end{bmatrix}^T$.



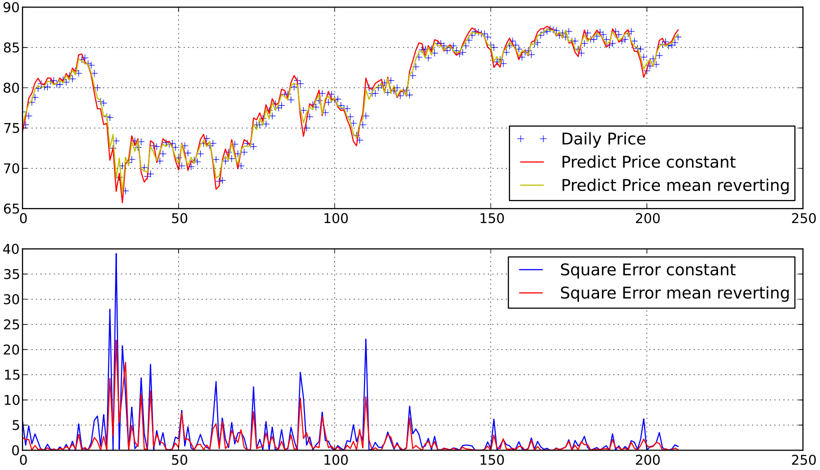
Mean-reverting Model - Application

- Use latest 20% data to recursively evaluate the $\hat{r}[n|n-1]$.



Constant versus Mean-reverting Model

- Mean-reverting model have better tracking error performance, especially when price change dramatically.



Summary

- The financial time series in real applications are always non-stationary. So there's no perfect model can fit them well.
- I assume the daily return series are stationary, and thus using two state space model (constant and time-reverting) to model it separately.
- Both models' parameters were estimated (analytically or numerically) through maximizing its likelihood function.
- Then based on the parameters, a configured Kalman filter is used to recursively predict and correct the underlying series.
- Not surprisingly, a more complicated mean-reverting model have better prediction performance than the constant one.

Reference

- Franco JCG. Maximum likelihood estimation of mean reverting processes. Real Options Practice. Onward Inc.
- Welch, Greg, and Gary Bishop. An Introduction to the Kalman Filter. Ed. L Hansson, L Fahrig, & G Merriam. Design 7.1 (2001) : 1-16.
- Paresh Date. Kalman Filtering in Mathematical Finance. CARISMA, Brunel University.
- Eric Zivot. Estimation of ARMA Models. 2005